

Cosmological solutions of massive gravity on de Sitter

David Langlois^{1,2}, Atsushi Naruko¹

¹ APC (CNRS-Université Paris 7), 10 rue Alice Domon et Léonie Duquet, 75205 Paris Cedex 13, France;

² IAP, 98bis Boulevard Arago, 75014 Paris, France

(Dated: July 10, 2012)

In the framework of the recently proposed models of massive gravity, defined with respect to a de Sitter reference metric, we obtain new homogeneous and isotropic solutions for arbitrary cosmological matter and arbitrary spatial curvature. These solutions can be classified into three branches. In the first two, the massive gravity terms behave like a cosmological constant. In the third branch, the massive gravity effects can be described by a time evolving effective fluid with rather remarkable features, including the property to behave as a cosmological constant at late time.

I. INTRODUCTION

Long after the first attempt by Pauli and Fierz to give a mass to the graviton [1], it has been realized, decades ago, that finding a healthy nonlinear massive extension of general relativity represents a formidable challenge because it requires to get rid of the so-called Boulware-Deser ghost [2]. Very recently, de Rham, Gabadadze and Tolley (dRGT) succeeded in constructing a massive theory of gravity that satisfies this criterion [3], as later confirmed in [4]. Beyond its obvious theoretical interest, this achievement has a special significance in a context where most of the matter content of the Universe remains unknown and alternative explanations for dark energy and/or dark matter could reveal appealing. This explains why this recent model has attracted a lot of attention, especially for its cosmological consequences. In this respect, a surprising discovery was that dRGT massive gravity does not allow for spatially *flat* homogeneous and isotropic solutions [5]. However, *open* cosmological solutions were obtained, with two branches of solutions in which the massive graviton terms lead to an effective cosmological constant [6] (other solutions relevant for cosmology can be found in e.g. [7–11]).

In the present work, we start from a slightly modified version of the original dRGT massive gravity in which the (a priori arbitrary) reference geometry is chosen to be de Sitter instead of Minkowski. A similar setting was explored very recently in [12] and [13]. The de Sitter geometry possesses as many symmetries as the flat geometry but introduces a mass scale H_c as additional parameter. In this setup, we have been able to find new cosmological solutions with flat, open or closed spatial geometry, for arbitrary cosmological matter. Our solutions can be classified in three branches, two of which being analogous to the open solutions of [6], while the last branch exhibits a new and rich phenomenology.

II. HOMOGENEOUS AND ISOTROPIC SOLUTIONS OF MASSIVE GRAVITY

We first present the theory of massive gravity introduced in [3], which can be described in terms of the usual four-dimensional metric $g_{\mu\nu}$ and of four scalar fields ϕ^a ($a = 0, \dots, 3$), called the Stückelberg fields. Gravity is governed by the action

$$S_{\text{grav}} = M_{\text{pl}}^2 \int d^4x \sqrt{-g} \left[\frac{1}{2} R + m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \right], \quad (2.1)$$

where the first term is the familiar Einstein-Hilbert Lagrangian (we set $M_{\text{pl}} = 1$ in the following) and the three additional terms are specific functions of the metric $g_{\mu\nu}$ and of the four scalar fields ϕ^a , via the tensor

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{f_{ab} g^{\mu\sigma} \partial_\sigma \phi^a \partial_\nu \phi^b}, \quad (2.2)$$

where f_{ab} is called the reference, or fiducial, metric (the square root must be understood in a matricial sense). The explicit expressions for these additional terms in the Lagrangian are

$$\mathcal{L}_2 = \frac{1}{2} ([\mathcal{K}]^2 - [\mathcal{K}^2]) \quad (2.3)$$

$$\mathcal{L}_3 = \frac{1}{6} ([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]) \quad (2.4)$$

$$\mathcal{L}_4 = \frac{1}{24} ([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]) \quad (2.5)$$

where the standard matrix notation is used (i.e. $(\mathcal{K}^2)^\mu_\nu = \mathcal{K}^\mu_\sigma \mathcal{K}^\sigma_\nu$) and the brackets represent a trace.

We now restrict our discussion to a FLRW (Friedmann-Lemaître-Robertson-Walker) geometry, of arbitrary spatial curvature, described by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) \gamma_{ij}(x) dx^i dx^j, \quad (2.6)$$

where the spatial metric γ_{ij} , written for example in terms of spherical coordinates, reads

$$\gamma_{ij}(x) dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2.7)$$

with $k = 0, -1$ or 1 for, respectively, flat, open or closed cosmologies.

In the present work, we take for the reference metric f_{ab} the de Sitter metric. As we will see, and in contrast with the Minkowski case, one can easily construct flat, open and closed cosmologies by starting from the appropriate slicing of de Sitter. Let us thus write the de Sitter metric in the form

$$f_{ab} d\phi^a d\phi^b = -dT^2 + b_k^2(T) \gamma_{ij}(X) dX^i dX^j, \quad (2.8)$$

where the functions $b_k(T)$ are defined by

$$b_0(T) = e^{H_c T}, \quad b_{-1}(T) = H_c^{-1} \sinh(H_c T), \quad b_1(T) = H_c^{-1} \cosh(H_c T). \quad (2.9)$$

In the limit $H_c \rightarrow 0$, one recovers the Minkowski metric in the flat and open cases: $b_0(T) = 1$ and $b_{-1}(T) = T$, the latter case corresponding to the Milne metric for the flat geometry.

We must now specify the Stückelberg fields so that the cosmological symmetries are satisfied. One sees immediately that the choice

$$\phi^0 = T = f(t), \quad \phi^i = X^i = x^i \quad (2.10)$$

leads to a homogeneous and isotropic tensor,

$$f_{\mu\nu} = f_{ab} \partial_\mu \phi^a \partial_\nu \phi^b = \text{Diag} \left\{ -\dot{f}^2, b_k^2(f(t)) \gamma_{ij} \right\}. \quad (2.11)$$

Denoting ε_f the sign of \dot{f} , the corresponding matrix \mathcal{K} , defined in (2.2), is simply given by¹

$$\mathcal{K}^0_0 = 1 - \varepsilon_f \frac{\dot{f}}{N}, \quad \mathcal{K}^i_j = \left(1 - \frac{b_k(f)}{a} \right) \delta^i_j, \quad \mathcal{K}^i_0 = 0, \quad \mathcal{K}^0_i = 0. \quad (2.12)$$

Substituting in the Lagrangian of massive gravity, one gets

$$\begin{aligned} \mathcal{L}_g &\equiv \sqrt{-g} (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \\ &= (a - b_k(f)) \left\{ N \left[a^2 (4\alpha_3 + \alpha_4 + 6) - a(5\alpha_3 + 2\alpha_4 + 3) b_k(f) + (\alpha_3 + \alpha_4) b_k^2(f) \right] \right. \\ &\quad \left. - \varepsilon_f \dot{f} \left[(3 + 3\alpha_3 + \alpha_4) a^2 - (3\alpha_3 + 2\alpha_4) a b_k(f) + \alpha_4 b_k(f)^2 \right] \right\}. \end{aligned} \quad (2.13)$$

The equation of motion for $f(t)$ is obtained by varying this Lagrangian with respect to f :

$$\left[(3 + 3\alpha_3 + \alpha_4) a^2 - 2(1 + 2\alpha_3 + \alpha_4) a b_k(f) + (\alpha_3 + \alpha_4) b_k^2(f) \right] \left(\frac{\dot{a}}{N} - \varepsilon_f b'_k(f) \right) = 0. \quad (2.14)$$

In general, there are several solutions for f . The first two solutions correspond to

$$b_k(f(t)) = X_\pm a(t), \quad X_\pm = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4}, \quad (2.15)$$

which exist only if the function b_k is invertible. For a Minkowski reference metric $f_{ab} = \eta_{ab}$, one sees immediatly that there is no solution in the flat case since $b_0(f) = 1$, whereas $b_{-1}(f) = f$ leads to two branches of solutions in the open case, in agreement with the conclusions of [5] and [6].

¹ We also assume $f > 0$ in the case $k = -1$.

Let us now concentrate on the last branch defined by the condition

$$\varepsilon_f b'_k(f) = \frac{\dot{a}}{N}. \quad (2.16)$$

It is non trivial only if b'_k is an invertible function, which is not the case with a Minkowski reference metric, either in the flat or open cases. However, in our case, one can obtain an explicit solution for $f(t)$ with the functions b_k given in (2.9). Before examining the flat case, let us stress that the solutions in this branch are necessarily accelerating as can be seen by taking the time derivative of (2.16), which yields

$$\ddot{a} = b''_k(f) |\dot{f}| > 0 \quad (N = 1). \quad (2.17)$$

In the particular case $k = 0$, on which we will focus in the following, one finds (assuming $\dot{f} > 0$)

$$f(t) = H_c^{-1} \ln \left(\frac{H(t) a(t)}{H_c} \right), \quad H \equiv \frac{\dot{a}}{Na} \quad (2.18)$$

where H denotes the usual Hubble parameter.

III. FRIEDMANN EQUATIONS AND EFFECTIVE GRAVITATIONAL FLUID

To obtain the Friedmann equations, one must add to \mathcal{L}_g the usual Einstein-Hilbert term, which reads

$$\mathcal{L}_{\text{EH}} = -\frac{3\dot{a}^2 a}{N} + 3kNa, \quad (3.1)$$

as well as an arbitrary matter Lagrangian \mathcal{L}_m that describes ordinary cosmological matter. Variation of the total Lagrangian with respect to the lapse function N (which will be set to 1 in the following) then yields the first Friedmann equation

$$3H^2 + 3\frac{k}{a^2} = \rho_m + \rho_g, \quad H \equiv \frac{\dot{a}}{a}, \quad (3.2)$$

where ρ_m denotes the ordinary matter energy density whilst ρ_g corresponds to an effective energy density arising from the massive gravity action:

$$\rho_g \equiv \frac{m_g^2}{a^3} (b_k(f) - a) \left\{ (6 + 4\alpha_3 + \alpha_4) a^2 - (3 + 5\alpha_3 + 2\alpha_4) a b_k(f) + (\alpha_3 + \alpha_4) b_k^2(f) \right\}. \quad (3.3)$$

The variation of the total action with respect to $a(t)$ yields the second Friedmann equation in the form

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -P_m - P_g, \quad (3.4)$$

with the effective pressure

$$P_g \equiv \frac{m_g^2}{a^3} \left\{ \left(6 + 4\alpha_3 + \alpha_4 - (3 + 3\alpha_3 + \alpha_4) \dot{f} \right) a^2 - 2 \left(3 + 3\alpha_3 + \alpha_4 - (1 + 2\alpha_3 + \alpha_4) \dot{f} \right) a b_k(f) + \left(1 + 2\alpha_3 + \alpha_4 - (\alpha_3 + \alpha_4) \dot{f} \right) b_k^2(f) \right\}. \quad (3.5)$$

We now study the expressions of ρ_g and P_g for the three branches of solutions identified previously.

1. First two branches

Substituting the solution (2.15), one finds that the massive gravity contribution behaves like a cosmological constant with

$$\rho_g = -P_g = -m_g^2 \frac{\left(1 + \alpha_3 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right) \left(1 + \alpha_3^2 - 2\alpha_4 \pm (1 + \alpha_3) \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right)}{(\alpha_3 + \alpha_4)^2}. \quad (3.6)$$

Note that the terms proportional to \dot{f} in (3.5) cancel because they are proportional to the combination that appears in the equation of motion for f . We recover exactly the result of [6], even if the spatial curvature is no longer restricted to be negative. Remarkably, the result is independent of H_c .

2. Third branch

Let us now turn to the third branch where the effective gravitational fluid follows a much more sophisticated behaviour. For simplicity, we consider here only the flat case, but it is straightforward to extend the following analysis to the open and closed cases. Upon substituting the explicit solution (2.18) for f into (3.3) and (3.5), one gets

$$\rho_g = -m_g^2 \left(1 - \frac{H}{H_c}\right) \left\{ 6 + 4\alpha_3 + \alpha_4 - (3 + 5\alpha_3 + 2\alpha_4) \frac{H}{H_c} + (\alpha_3 + \alpha_4) \frac{H^2}{H_c^2} \right\} \quad (3.7)$$

and

$$P_g = m_g^2 \left\{ 6 + 4\alpha_3 + \alpha_4 - (3 + 3\alpha_3 + \alpha_4) \frac{H}{H_c} \left(3 + \frac{\dot{H}}{H^2} \right) + (1 + 2\alpha_3 + \alpha_4) \frac{H^2}{H_c^2} \left(3 + 2 \frac{\dot{H}}{H^2} \right) - (\alpha_3 + \alpha_4) \frac{H^3}{H_c^3} \left(1 + \frac{\dot{H}}{H^2} \right) \right\}. \quad (3.8)$$

It can be noticed that (3.7) coincides with the expression obtained by [13] in the special case of de Sitter cosmology, i.e. with a constant H , on a de Sitter reference metric, although the Hubble parameter is time-dependent in our case. One can check explicitly that the effective gravitational fluid, characterized by ρ_g and P_g , satisfies the usual conservation equation

$$\dot{\rho}_g + 3H(\rho_g + P_g) = 0. \quad (3.9)$$

The behaviour of the effective fluid described by the above energy density and pressure is quite peculiar. The energy density ρ_g can be positive or negative and its sign can change during time evolution, when the ratio H/H_c crosses specific values, which depend on the parameters α_3 and α_4 . For example, in the minimal model with $\alpha_3 = \alpha_4 = 0$, ρ_g changes sign when $H = H_c$ or $H = 2H_c$.

Combining the second Friedmann equation with the first one yields the relation

$$\left[m_g^2 \frac{H}{H_c} \left(3 + 3\alpha_3 + \alpha_4 - 2(1 + 2\alpha_3 + \alpha_4) \frac{H}{H_c} + (\alpha_3 + \alpha_4) \frac{H^2}{H_c^2} \right) - 2H^2 \right] \frac{\dot{H}}{H^2} = (1 + w_m) \rho_m, \quad (3.10)$$

where $w_m \equiv P_m/\rho_m$. One can thus identify a critical value for H when the term between brackets, of the form $\tilde{m}^2 - 2H^2 \equiv m_g^2 \chi(H/H_c) - 2H^2$, vanishes. Remarkably, this critical value coincides with the Higuchi bound derived in [13] for de Sitter cosmology. Moreover, one can see that for the two other branches discussed earlier, the function $\chi(H/H_c)$ vanishes, which seems to be related to the fact that the kinetic energy of the scalar mode around these solutions vanishes, as found in [14], since the kinetic term is proportional to $\tilde{m}^2(\tilde{m}^2 - 2H^2)$ according to [13]. If the Higuchi condition $\tilde{m}^2 > 2H^2$ is satisfied, the above relation implies that the Hubble parameter increases for matter satisfying the weak energy condition (i.e. $w_m > -1$ and $\rho_m > 0$).

Let us try to analyze the combined evolution of the effective gravitational fluid with ordinary matter, in the minimal model where $\alpha_3 = \alpha_4 = 0$ for simplicity. It is convenient to introduce the dimensionless quantities

$$Y \equiv \frac{H}{H_c}, \quad r \equiv \frac{\rho_m}{H_c^2}, \quad \lambda \equiv \frac{m_g^2}{H_c^2}. \quad (3.11)$$

Using the first Friedmann equation to express ρ_m as a function of H , one finds that the above equation can be rewritten as a differential equation for Y and the full system can be written in the form

$$Y' = \frac{3(1 + w_m) [(1 + \lambda)Y^2 - 3\lambda Y + 2\lambda]}{3\lambda - 2(1 + \lambda)Y}, \quad r = 3 [(1 + \lambda)Y^2 - 3\lambda Y + 2\lambda] \quad (3.12)$$

where a prime denotes a derivative with respect to the number of e-folds, i.e. $\dot{Y} = HY'$. The second relation is simply a constraint between the value of the matter energy density and the Hubble parameter. In the following, we will only assume that the cosmological matter is characterized by $r > 0$ and $w_m > -1$. It is then worth noting that the Higuchi condition $\tilde{m}^2 - 2H^2 > 0$ corresponds to

$$\mathcal{H} \equiv 3\lambda - 2(1 + \lambda)Y = -2(1 + \lambda)(Y - Y_H) > 0, \quad Y_H \equiv \frac{3\lambda}{2(1 + \lambda)}. \quad (3.13)$$

In order to satisfy the Higuchi bound, one must therefore have $Y < Y_H$ if $\lambda > -1$, or $Y > Y_H$ if $\lambda < -1$.

It is also useful to introduce the two roots of the numerator of the equation for Y ,

$$Y_{\pm} = \frac{3\lambda \pm \sqrt{\lambda(\lambda - 8)}}{2(1 + \lambda)}, \quad (3.14)$$

which are defined if $\lambda > 8$ or $\lambda < 0$. Rewriting the dynamical system (3.12) in the form

$$Y' = -\frac{3}{2}(1 + w_m) \frac{(Y - Y_+)(Y - Y_-)}{Y - Y_H}, \quad r = 3(1 + \lambda)(Y - Y_+)(Y - Y_-) > 0, \quad (3.15)$$

it is easy to study its evolution, depending on the value of λ :

- $\lambda > 8$ (which implies $0 < Y_- < Y_H < Y_+$): if $\mathcal{H} > 0$, then $Y < Y_-$ and Y tends towards Y_- asymptotically. By contrast, if the Higuchi bound is not satisfied, i.e. $\mathcal{H} < 0$, one must have $Y > Y_+$ and Y decreases, converging asymptotically towards Y_+ .
- $0 < \lambda < 8$ (Y_+ and Y_- are not defined): the Higuchi bound is satisfied if $Y < Y_H$ initially, and Y increases to reach Y_H in a *finite* time. By contrast, if $Y > Y_H$ initially, the Higuchi bound is not satisfied and Y decreases to reach Y_H in a finite time.
- $-1 < \lambda < 0$ ($Y_H < 0$): the Higuchi condition is never satisfied. The condition $r > 0$ imposes $Y > Y_+$ and Y decreases toward Y_+ asymptotically.
- $\lambda < -1$ (which implies $0 < Y_+ < Y_H < Y_-$): $\mathcal{H} > 0$ imposes $Y_H < Y < Y_-$ initially and Y tends asymptotically towards Y_- . If $\mathcal{H} < 0$, one must have $Y_+ < Y < Y_H$ and Y decreases towards Y_+ .

We thus find that in most cases ($\lambda < 0$ or $\lambda > 8$), the effective gravitational energy density tends to a constant asymptotically, while the cosmological evolution approaches de Sitter, with a Hubble parameter that depends on λ and is proportional to H_c . When $0 < \lambda < 8$, the system evolves towards a singularity at finite time. One can proceed similarly for general values α_3 and α_4 but the analysis is more complicated because the numerator and denominator of the equation for Y become respectively third-order and second-order polynomials in Y .

IV. CONCLUSION

In the present work, we have obtained spatially flat (as well as open or closed) FLRW solutions with arbitrary cosmological matter in the context of ghost-free models of massive gravity, evading the no-go theorem of [5] by adopting a de Sitter reference metric instead of Minkowski. The constraint equation for the Stückelberg fields leads to three branches. In two branches, one finds that the effective gravitational fluid behaves like a cosmological constant, whose value, remarkably, is independent of H_c and coincides exactly with the value obtained in [6] for the specific case of open FLRW solutions with Minkowski as reference metric. By contrast, the third branch exhibits a much richer phenomenology, although expanding cosmological solutions are restricted to be accelerating. The massive gravity effects can be described by an effective fluid, which is in general time-dependent since its energy density depends on the physical Hubble parameter H (and its pressure on H as well). In the simplest case where $\alpha_3 = \alpha_4 = 0$ we have investigated the cosmological evolution and found that the outcome is either a singularity at finite time or a de Sitter evolution, depending on the value of the ratio $\lambda = m_g^2/H_c^2$.

To conclude, massive gravity on de Sitter leads to new solutions with surprising features. It would be worth exploring further these solutions, in particular by investigating more systematically the parameter space for the coefficients α_3 and α_4 . It would also be interesting to study perturbations around these new solutions, by extending previous works on this topic (see e.g. [13, 14]).

Acknowledgments

We would like to thank Shinji Mukohyama for very instructive discussions, as well as Matteo Fiasello, Kazuya Koyama, Eric Linder and Andrew Tolley for helpful remarks. A.N. was supported by JSPS Postdoctoral Fellowships

for Research Abroad. D.L. was partly supported by the ANR (Agence Nationale de la Recherche) grant STR-COSMO ANR-09-BLAN-0157-01.

-
- [1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939).
 - [2] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
 - [3] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].
 - [4] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012) [arXiv:1106.3344 [hep-th]].
 - [5] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, Phys. Rev. D **84**, 124046 (2011) [arXiv:1108.5231 [hep-th]].
 - [6] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP **1111**, 030 (2011) [arXiv:1109.3845 [hep-th]].
 - [7] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. Lett. **107**, 131101 (2011) [arXiv:1103.4708 [hep-th]].
 - [8] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. D **84**, 064033 (2011) [arXiv:1104.2143 [hep-th]].
 - [9] P. Gratia, W. Hu and M. Wyman, arXiv:1205.4241 [hep-th].
 - [10] T. Kobayashi, M. Siino, M. Yamaguchi and D. Yoshida, arXiv:1205.4938 [hep-th].
 - [11] M. S. Volkov, arXiv:1205.5713 [hep-th].
 - [12] C. de Rham and S. Renaux-Petel, arXiv:1206.3482 [hep-th].
 - [13] M. Fasiello and A. J. Tolley, arXiv:1206.3852 [hep-th].
 - [14] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP **1203**, 006 (2012) [arXiv:1111.4107 [hep-th]].